On a Class of Electromagnetic Wave Functions for Propagation Along the Circular Gyrotropic Waveguide

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Abstract —The properties of confluent hypergeometric functions as exact electromagnetic wave functions for propagation in a circular waveguide containing azimuthally magnetized remanent ferrite are investigated. Two different forms of solutions of the propagation problem for angular symmetric transverse electric modes are constructed—one in terms of Kummer and Tricomi confluent hypergeometric functions of complex parameter and variable and a second in terms of Whittaker functions. An evaluation of this class of wave functions is performed to sufficient extent, followed by tabulation of their imaginary zeros, providing computation of eigenvalue spectrum and phase characteristics of the gyrotropic guide.

I. Introduction

THE problem of finding exact wave functions for propagation in anisotropic cylindrically stratified media is still of considerable interest. A related subject is the study of wave propagation along the circular waveguide, loaded with ferrite or semiconductor, made anisotropic by the application of an external azimuthal dc magnetic field [1]–[11]. Due to the tensor character of magnetic permeability or permittivity of such media, the solution of propagation problems leads to a system of second-order partial differential equations for the longitudinal components of electric and magnetic vectors. A direct solution of the electromagnetic field equations becomes possible in the degenerate case of angular symmetry only. The relevant equations for modes exhibiting nonreciprocal properties in such gyrotropic guiding structures are found to be Kummer [11]–[14] or Whittaker [1], [2], [10] forms of the confluent hypergeometric equation. The procedure of finding the fundamental system of solutions of these equations either in terms of Kummer and Tricomi functions or in terms of Whittaker first and second functions is well established [15]–[17], [20]. Nevertheless, many authors have avoided these functions because of concern over their zeros and insufficient tabulation [3], [6], [8].

This paper deals with the class of confluent hypergeometric functions (CHF) of complex parameter and variable as exact wave functions for angular symmetric TE-mode propagation along a circular waveguide, filled with ferrite, magnetized azimuthally to remanence by a coaxial switching conductor of finite radius. Two different forms of the exact solution of the propagation problem are constructed: one in terms of Kummer and Tricomi functions and a second in terms of Whittaker first and second functions. The properties of these functions are discussed to sufficient extent. The transcendental characteristic equation of the structure derived in terms of CHF permits us to draw conclusions about the nonreciprocal character of wave propagation.

To facilitate the analysis, the switching conductor is assumed infinitely thin, which reduces the guide cross section into a simply connected region. The resulting structure is a canonical one for comprehensive study of Kummer and Whittaker first functions, both being regular in the whole region. Detailed tables of the wave functions and their zeros, compiled for the first time, allow one to predict the phase characteristics of the gyrotropic guide. The practical usefulness of the proposed exact analysis of wave propagation stems also from the possibility of obtaining a simplified asymptotic solution without cumbersome numerical computation [14].

The study of this class of electromagnetic wave functions is of particular interest in exploring a variety of radially stratified guiding structures, containing azimuthally magnetized ferrite or solid plasma, under angular symmetric wave excitation.

II. Formulation of the Problem

The structure to be considered is an infinitely long perfectly conducting circular waveguide of radius r_0 loaded with latching ferrite, magnetized azimuthally to remanence by a coaxially positioned switching conductor of radius r_1 (Fig. 1), propagating fast electromagnetic waves. An appropriate cylindrical coordinate system (r, θ, z) is adopted with the z-axis along the geometric axis of the guide. The remanent ferrite is characterized by a scalar lossless permittivity $\epsilon = \epsilon_0 \epsilon_r$ and a tensor permeability of the Polder form

$$\overleftrightarrow{\mu} = \mu_0 \begin{bmatrix} 1 & 0 & -j\alpha \\ 0 & 1 & 0 \\ j\alpha & 0 & 1 \end{bmatrix}$$

where $\alpha = p = \gamma M_r/\omega$, γ is the gyromagnetic ratio, M_r is the ferrite remanent magnetization, ω is the angular frequency of the propagating wave, and ϵ_0, μ_0 are the free

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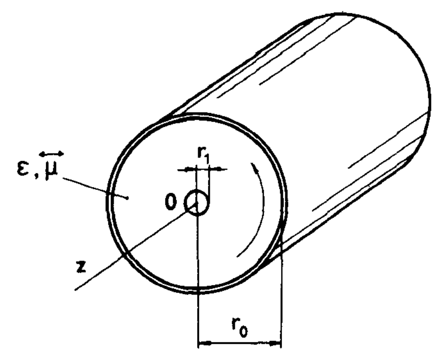


Fig. 1. Geometry of the problem.

space permittivity and permeability, respectively. We consider the whole ferrite medium magnetized to remanence M_r irrespective of the distance from the switching conductor. A positive or negative sign is ascribed to M_r, corresponding to its counterclockwise or clockwise direction with respect to the direction of wave propagation.

III. Mathematical Treatment

Expressing the electric and magnetic vectors as column vectors and writing the curl operator in matrix form, the Maxwell equations in the ferrite medium for harmonic time dependence $\exp(j\omega t)$ take the form

$$\begin{bmatrix} 0 & -D_z & r^{-1}D_\theta \\ D_z & 0 & -D_r \\ -r^{-1}D_\theta & r^{-1}D_r r & 0 \end{bmatrix}\begin{bmatrix} H_r \\ H_\theta \\ H_z \end{bmatrix} = j\omega\epsilon_0\epsilon_r \begin{bmatrix} E_r \\ E_\theta \\ E_z \end{bmatrix} \tag{1}$$

$$\begin{bmatrix} 0 & -D_z & r^{-1}D_\theta \\ D_z & 0 & -D_r \\ -r^{-1}D_\theta & r^{-1}D_r r & 0 \end{bmatrix}\begin{bmatrix} E_r \\ E_\theta \\ E_z \end{bmatrix} = -j\omega\mu_0 \begin{bmatrix} 1 & 0 & -j\alpha \\ 0 & 1 & 0 \\ j\alpha & 0 & 1 \end{bmatrix}\begin{bmatrix} H_r \\ H_\theta \\ H_z \end{bmatrix} \tag{2}$$

where $D_r = \partial/\partial r$, $D_\theta = \partial/\partial\theta$, $D_z = \partial/\partial z$ are differential operators.

Assuming a z-dependence of the form $\exp(-j\beta z)$, $D_z = -j\beta$, where β is the phase constant of the incident wave propagating in the direction of increasing z, and eliminating successively the transverse field components from (1) and (2), the following system of two partial differential equations for the longitudinal components of the electric and magnetic vector is obtained:

$$\begin{aligned} &\left(\frac{1}{r}D_r r D_r + \beta_1^2 - \beta^2 + \frac{1}{r^2}D_\theta^2\right)E_z + \omega\mu_0\alpha\frac{1}{r}D_\theta H_z = 0 \\ &\omega\epsilon_0\epsilon_r\alpha\frac{1}{r}D_\theta E_z - \left(\frac{1}{r}D_r r D_r + \beta_f^2 - \beta^2 - \alpha\beta\frac{1}{r} + \frac{1}{r^2}D_\theta^2\right)H_z = 0 \end{aligned} \tag{3}$$

with $\beta_1^2 = \omega^2\epsilon_0\mu_0\epsilon_r$, $\beta_f^2 = \omega^2\epsilon_0\mu_0\epsilon_r\mu_{\text{eff}}$, $\mu_{\text{eff}} = 1-\alpha^2$ being the effective relative permeability of azimuthally magnetized ferrite.

The expressions

$$E_r = -\frac{j}{\beta_1^2-\beta^2}\left(\beta D_r E_z + \frac{\omega\mu_0}{r}D_\theta H_z\right) \tag{4}$$

$$H_\theta = -j\frac{\omega\epsilon_0\epsilon_r}{\beta_1^2-\beta^2}\left(D_r E_z + \frac{1}{\omega\epsilon_0\epsilon_r}\frac{\beta}{r}D_\theta H_z\right) \tag{5}$$

$$E_\theta = -\frac{j}{\beta_1^2-\beta^2}\left(\frac{\beta}{r}D_\theta E_z - \omega\mu_0 D_r H_z + \omega\mu_0\alpha\beta H_z\right) \tag{6}$$

$$H_r = j\frac{\omega\epsilon_0\epsilon_r}{\beta_1^2-\beta^2}\left(\frac{1}{r}D_\theta E_z - \frac{\beta}{\omega\epsilon_0\epsilon_r}D_r H_z + \omega\mu_0\alpha H_z\right) \tag{7}$$

permit us to find the transverse field components of the propagating wave whenever the longitudinal components E_z and H_z are known.

Restricting the discussion to angular symmetric fields ($D_\theta = 0$), the propagation problem can be formulated in terms of a single longitudinal component of the electric or magnetic field [12]–[14]. Referring to (4)–(7) the field decomposes into TM (E_r, H_θ, E_z) and TE (H_r, E_θ, H_z) modes depending on which longitudinal component is present. In this degenerate case the system (3) splits into two independent second-order ordinary differential equations for E_z and H_z

$$\left(\frac{1}{r}D_r r D_r + \beta_1^2 - \beta^2\right)E_z = 0 \tag{8}$$

$$\left(\frac{1}{r}D_r r D_r + \beta_f^2 - \beta^2 - \alpha\beta\frac{1}{r}\right)H_z = 0. \tag{9}$$

The substitution $\Gamma_f = (\beta_1^2 - \beta^2)^{1/2}$, Γ_f being the radial wavenumber, transforms (8) into a zeroth-order Bessel equation. Since the ferrite-filled region does not include the origin $r = 0$, the general solution for E_z should be written in the form

$$E_z = AJ_0(\Gamma_f r) + BN_0(\Gamma_f r) \tag{10}$$

where J_0, N_0 are zeroth-order Bessel and Neumann functions and A, B are arbitrary constants.

By appropriate transformation of variables $x = j2\beta_2 r = jz$, $\beta_2 = (\beta_f^2 - \beta^2)^{1/2}$ being the radial wavenumber, and $H_z = y(x)e^{-x/2}$, (9) reduces to the Kummer form of the confluent hypergeometric equation [15], [16]

$$x\frac{d^2y}{dx^2} + (c-x)\frac{dy}{dx} - ay = 0 \tag{11}$$

with $c = 1$, $a = 0.5 - jk$, and $k = \alpha\beta/2\beta_2$. This equation has regular and irregular singularities at 0 and ∞, respectively.

The fundamental system of solutions of (11) in the double-connected ferrite-filled region excluding the origin $r = 0$, for c a positive integer and a neither a negative integer nor zero, is expressed in terms of Kummer and Tricomi CHF Φ and Ψ, respectively [15], [16]:

$$y = C\Phi(a, c; x) + D\Psi(a, c; x) \tag{12}$$

where C, D are arbitrary constants. The Kummer CHF is defined by the series

$$\Phi(a,c;x)=\sum_0^\infty \frac{(a)_\nu}{(c)_\nu}\cdot\frac{x^\nu}{\nu!} \quad (13)$$

which is absolutely convergent for all values (real or complex) of a, c, x except $c=0,-1,-2,-3,\cdots$. The Pochhammer's symbol $(\lambda)_\nu$, where λ denotes any number (real or complex) and ν denotes any positive integer or zero, is determined by the relation [19]

$$(\lambda)_\nu=\lambda(\lambda+1)(\lambda+2)\cdot\cdots\cdot(\lambda+\nu-1)=\frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}$$

with Γ being the gamma function. In particular, $(\lambda)_0=1$ and $(1)_\nu=\nu!$. Φ is analytic, i.e., single-valued and differentiable with respect to all x (real or complex). It is also analytic of a, but not of $c=0,-1,-2,-3,\cdots$, for which it has simple poles.

The Tricomi CHF is a multiple-valued function for which the zero is a branch point, with the main branch determined by the condition $-\pi<\arg x\leqslant\pi$. For $c=l+1$ ($l=0,1,2,\cdots$), it is given by the expression [15], [16]

$$\Psi(a,l+1;x)=\frac{(-1)^{l+1}}{l!\Gamma(a-l)}\Bigg\{\Phi(a,l+1;x)\ln x + \sum_0^\infty\frac{(a)_\nu}{(l+1)_\nu}\cdot\frac{x^\nu}{\nu!}[\psi(a+\nu) - \psi(1+\nu)-\psi(1+l+\nu)]\Bigg\} + \frac{(l-1)!}{\Gamma(a)}\sum_0^{l-1}\frac{(a-l)_\nu}{(1-l)_\nu}\cdot\frac{x^{\nu-l}}{\nu!}. \quad (14)$$

The last sum in (14) vanishes for $l=0$. The function $\psi(x)=\Gamma'(x)/\Gamma(x)$ is the logarithmic derivative of the gamma function. As seen from (14), Ψ is defined for all values (real or complex) of a, c, x, except $x=0$.

Another approach for solution of the propagation problem is based on the transformations $x=j2\beta_2 r=jz$, $H_z=w(x)x^{-1/2}$, $\kappa=j\alpha\beta/2\beta_2=jk$, which modifies (9) for H_z to the special case ($m=0$) of Whittaker form of the confluent hypergeometric equation [16], [17], [20]

$$\frac{d^2w}{dx^2}+\left(-\frac{1}{4}+\frac{\kappa}{x}+\frac{1-4m^2}{4x^2}\right)w=0. \quad (15)$$

This equation could also be obtained directly from (11) under the transformations $w(x)=y(x)x^{c/2}e^{-x/2}$, $\kappa=(c/2)-a$, $m=(c-1)/2$. The fundamental system of solutions of (15) is expressed in terms of Whittaker first and second functions $M_{\kappa,m}(x)$ and $W_{\kappa,m}(x)$, respectively

$$w(x)=FM_{\kappa,m}(x)+GW_{\kappa,m}(x) \quad (16)$$

F, G being arbitrary constants. The Whittaker first function $M_{\kappa,m}(x)$ is defined by the infinite series [17]

$$M_{\kappa,m}(x)=x^{1/2+m}e^{-x/2}\sum_0^\infty\frac{\left(\frac{1}{2}+m-\kappa\right)_\nu}{(2m+1)_\nu}\cdot\frac{x^\nu}{\nu!}. \quad (17)$$

The Whittaker second function $W_{\kappa,m}(x)$ for $2m$ an integer is given by the expression [16]

$$W_{\kappa,m}(x)=\frac{(-1)^{2m+1}M_{\kappa,m}(x)\ln x}{\Gamma\left(\frac{1}{2}-m-\kappa\right)\cdot\Gamma(1+2m)} + \frac{(-1)^{2m+1}x^{1/2+m}e^{-x/2}}{\Gamma\left(\frac{1}{2}-m-\kappa\right)}\cdot\Bigg\{\sum_0^\infty\frac{\left(\frac{1}{2}+m-\kappa\right)_\nu}{(1)_{2m+\nu}} \cdot\frac{x^\nu}{\nu!}\left[\psi\left(\tfrac{1}{2}+m-\kappa+\nu\right)-\psi(1+\nu) - \psi(1+2m+\nu)\right] - \sum_1^{2m}\frac{(\nu-1)!x^{-\nu}}{(1)_{2m-\nu}\cdot\left(\frac{1}{2}-m+\kappa\right)_\nu}\Bigg\}. \quad (18)$$

For complex x, both $M_{\kappa,m}(x)$ and $W_{\kappa,m}(x)$ are multiple valued in the complex x plane. The origin $x=0$ is a branch point, and the point $x=\infty$ is an essential singularity point for these functions. Since $|\arg x|<\pi$, the principle branch of Whittaker CHF is considered [16], [17], [20]. In addition, $M_{\kappa,m}(x)$ is analytic for all values of κ, m, and x, provided that $m=-(2l+1)/2$ ($l=0,1,2,3,\cdots$) and $W_{\kappa,m}(x)$—for all κ, m, x, except $x=0$. Both $\Phi(a,c;x)$ and $M_{\kappa,m}(x)$ are regular at zero, whereas $\Psi(a,c;x)$ and $W_{\kappa,m}(x)$ tend to infinity for $x\to 0$.

The transformation of (9) to (11) or (15) yields the following expressions for H_z:

$$H_z=\left[C\Phi(0.5-jk,1;j2\beta_2 r) + D\Psi(0.5-jk,1;j2\beta_2 r)\right]e^{-j\beta_2 r} \quad (19)$$

$$H_z=\left[FM_{\kappa,0}(j2\beta_2 r)+GW_{\kappa,0}(j2\beta_2 r)\right](j2\beta_2 r)^{-1/2}. \quad (20)$$

The character of CHF as exact wave functions for propagation of angular symmetric TE modes along the circular gyrotropic guide becomes evident taking into account (4)–(7), (19), and (20).

To conclude this section, we list the relations between the various CHF which could be used to verify the results of the analysis [15]

$$M_{\kappa,m}(x)=x^{1/2+m}e^{-x/2}\Phi\left(\tfrac{1}{2}+m-\kappa,1+2m;x\right) \quad (21)$$

$$W_{\kappa,m}(x)=x^{1/2+m}e^{-x/2}\Psi\left(\tfrac{1}{2}+m-\kappa,1+2m;x\right). \quad (22)$$

IV. Transverse Field Components

The transverse angular symmetric field components are readily obtained from (4)–(7), (10), (19), and (20), using the derivative formulas and recurrence relations for the cylindrical functions and CHF [16], [19]:

$$E_r=(j\beta/\Gamma_f)\left[AJ_1(\Gamma_f r)+BN_1(\Gamma_f r)\right] \quad (23)$$

$$H_\theta=(j\omega\epsilon_0\epsilon_r/\Gamma_f)\left[AJ_1(\Gamma_f r)+BN_1(\Gamma_f r)\right] \quad (24)$$

$$E_\theta=\left(j\omega\mu_0/\Gamma_f^2\right)4\beta_2^2(0.5-jk)r \cdot\left[-0.5(0.5+jk)C\Phi(1.5-jk,3;j2\beta_2 r) + D\Psi(1.5-jk,3;j2\beta_2 r)\right]e^{-j\beta_2 r} \quad (25)$$

$$H_r = j\Big\{C\big[0.5(1-\alpha^2)\beta r\Phi(1.5-jk,3;\,j2\beta_2 r) + \alpha\Phi(0.5-jk,1;\,j2\beta_2 r)\big] + D\Big[-\frac{(1-\alpha^2)\beta r}{0.5+jk}\Psi(1.5-jk,3;\,j2\beta_2 r) + \alpha\Psi(0.5-jk,1;\,j2\beta_2 r)\Big]\Big\}e^{-j\beta_2 r}. \tag{26}$$

Alternately, the components E_θ, H_r could be expressed in terms of the Whittaker CHF

$$E_\theta = \left(\omega\mu_0/\Gamma_f^2\right)2\beta_2(0.5-\kappa)\cdot\big[-0.5(0.5+\kappa)FM_{\kappa,1}(j2\beta_2 r) + GW_{\kappa,1}(j2\beta_2 r)\big](j2\beta_2 r)^{-1/2} \tag{27}$$

$$H_r = \Big\{F\big[(1-\alpha^2)(\beta/4\beta_2)M_{\kappa,1}(j2\beta_2 r) + j\alpha M_{\kappa,0}(j2\beta_2 r)\big] + G\Big[-\frac{1-\alpha^2}{0.5+\kappa}(\beta/2\beta_2)W_{\kappa,1}(j2\beta_2 r) + j\alpha W_{\kappa,0}(j2\beta_2 r)\Big]\Big\}(j2\beta_2 r)^{-1/2}. \tag{28}$$

An inspection of (10), (23), and (24) reveals that the TM set of fields (E_r, H_θ, E_z) expressed in terms of Bessel and Neumann functions is independent of remanent ferrite magnetic parameters. Since the only magnetic field component H_θ is everywhere parallel with the remanent magnetization M_r, no interaction is possible between the TM mode and the ferrite. Thus, the angular symmetric TM mode propagates along the gyrotropic guide as in a circular guide filled with isotropic dielectric of permittivity ϵ_r.

Of major interest is the fast wave propagation of the angular symmetric TE (H_r, E_θ, H_z) mode expressed either in terms of Kummer and Tricomi, or in terms of Whittaker functions of complex parameter and variable (β_2 real). This mode exhibits a strong dependence on the sign and magnitude of M_r as seen from the parameters of all wave functions. Consequently, a change of field pattern and power handling capability of the guide under angular symmetric TE mode excitation should be expected when the ferrite remanent magnetization is reversed in the azimuthal direction.

The expressions of the field components are simplified considerably if the switching conductor becomes infinitely thin. In this case the anisotropic ferrite region includes the origin. To meet the requirements for finite fields, we put the arbitrary constants $B = D = G = 0$, yielding simpler expressions of field components in terms of Bessel, Kummer, and first Whittaker functions, all regular at $r = 0$.

Equations (21) and (22) permit an easy transition between the expressions of field components in terms of different CHF.

V. Characteristic Equations

Applying the boundary conditions at the interfaces $r = r_0$ and $r = r_1$, we obtain the following transcendental characteristic equations for TM and TE modes, respectively:

$$\frac{J_0(\Gamma_f r_0)}{N_0(\Gamma_f r_0)} = \frac{J_0(\Gamma_f r_1)}{N_0(\Gamma_f r_1)} \tag{29}$$

$$\frac{\Phi(1.5-jk,3;\,j2\beta_2 r_0)}{\Psi(1.5-jk,3;\,j2\beta_2 r_0)} = \frac{\Phi(1.5-jk,3;\,j2\beta_2 r_1)}{\Psi(1.5-jk,3;\,j2\beta_2 r_1)}. \tag{30}$$

Following the same procedure or substituting (21) and (22) into (30), the following characteristic equation for angular symmetric TE modes could also be obtained in terms of Whittaker functions:

$$\frac{M_{\kappa,1}(j2\beta_2 r_0)}{W_{\kappa,1}(j2\beta_2 r_0)} = \frac{M_{\kappa,1}(j2\beta_2 r_1)}{W_{\kappa,1}(j2\beta_2 r_1)}. \tag{31}$$

Equation (29) is easily recognized as the characteristic equation of the angular symmetric TM mode in the coaxial waveguide filled with isotropic dielectric of permittivity ϵ_r.

We now concentrate on the two forms of the characteristic equation for the angular symmetric TE mode (30) and (31), written in terms of CHF. Both characteristic equations involve implicitly the phase constant, parameters of the gyrotropic medium, structure geometry, and frequency.

The second Kummer theorem [16] (cf. the Appendix), for $k = 0$, $2a$ neither a zero nor a negative integer and $2m$ not a negative integer, facilitates checking the results of analysis against the case of angular symmetric TE mode propagation in the dielectric-loaded waveguide. Introducing $\alpha = 0$ ($k = 0$, $\beta_2 \equiv \Gamma_f$) in (30) and (31), the well-known equation for the dielectric-loaded coaxial guide is obtained as follows:

$$\frac{J_1(\Gamma_f r_0)}{N_1(\Gamma_f r_0)} = \frac{J_1(\Gamma_f r_1)}{N_1(\Gamma_f r_1)}. \tag{32}$$

One important subcase is of special interest. Assuming an infinitely thin switching conductor ($r_1 = 0$), the guide cross section transforms to a simply connected region, containing the regular singularity point $r = 0$ of Neumann, Tricomi, and Whittaker second functions. In view of this, the characteristic equations (29)–(31) take the following simpler forms:

$$J_0(\Gamma_f r_0) = 0 \tag{33}$$

$$\Phi(1.5-jk,3;\,j2\beta_2 r_0) = 0 \tag{34}$$

$$M_{\kappa,1}(j2\beta_2 r_0) = 0. \tag{35}$$

The circular gyrotropic guide with an infinitely thin switching conductor is a canonical structure for rigorous study of the eigenfunctions $\Phi(a,c;x)$ and $M_{\kappa,1}(x)$, regular at zero, and their eigenvalues. As seen from (17) or (21), the zeros $\zeta_{k,n}$ of Φ and M coincide since the factor $x^{(1/2)+m}\cdot e^{-x/2}$ has no zeros. The characteristic equations (34) and (35) require simply that $2\beta_2 r_0 = \zeta_{k,n}$, $\zeta_{k,n}$ being the nth consecutive zero of the Kummer and Whittaker first function, for given k ($\kappa = jk$), $n = 1,2,3,\cdots$. This restricts the radial wavenumber β_2 to have one of the following discrete values:

$$\beta_2 = \zeta_{k,n}/2r_0 \tag{36}$$

which define the eigenvalue spectrum of the angular symmetric TE mode.

The analysis shows the existence of two phase constants β_+ and β_- of the propagating $TE_{0,n}$ mode for $+k$ and $-k$

$$\beta_\pm = \left[\omega^2\epsilon_0\mu_0\epsilon_r\mu_{\text{eff}} - (\zeta_{\pm k,n}/2r_0)^2\right]^{1/2} \tag{37}$$

determined by the two stable states of remanent magnetization.

In what follows, we study the properties of wave functions to provide an exact numerical solution of the characteristic equations (34) and (35) and computation of the phase characteristics of the gyrotropic guide. For an approximate approach to the solution of (34), based on the asymptotic expansion of exact wave function $\Phi(a,c;x)$, the reader is referred to [14].

VI. Properties of Electromagnetic Wave Functions

The exact solution of the propagation problem in the circular gyrotropic waveguide with infinitely thin switching conductor requires detailed knowledge of the Kummer or Whittaker first CHF of complex parameter and variable. The eigenvalue analysis needs the finding of imaginary zeros of the wave functions, which provide computation of phase characteristics of the guide. In view of this, a detailed numerical evaluation of $\Phi(1.5-jk,3;jz)$ and $M_{jk,1}(jz)$ has been accomplished for the first time, followed by extensive study of their descriptive properties and a tabulation of their imaginary zeros.

Fig. 2 is a plot of loci curves of $\Phi(1.5-jk,3;jz)$ in the complex plane for $k=+0.1$ (solid curve) and $k=-0.1$ (dashed curve), while Fig. 3 shows the variations of $\text{Re}\,\Phi$ (solid curves) and $\text{Im}\,\Phi$ (dashed curves) versus z for $k = 0, \pm 0.1, \pm 0.3, \pm 0.5$. The points of intersection of $\text{Re}\,\Phi$ and $\text{Im}\,\Phi$ with coordinate axes should be examined more closely. As seen from Fig. 3, all curves of the family $\text{Re}\,\Phi$ start from unity at the vertical axis and intersect the horizontal one at the points $z=(2n-1)\pi$, $(n=1,2,3,\cdots)$ for all k. The lines $\text{Im}\,\Phi$ beginning at zero intersect the z-axis at points $z=(n-1)2\pi$, irrespective of the value of k. These features of $\text{Re}\,\Phi$ and $\text{Im}\,\Phi$ could easily be obtained analytically from the first Kummer theorem [16]. Curves of both families, corresponding to the same k, intersect simultaneously the horizontal axis at points $\zeta_{k,n}$, depicting the consecutive zeros of Φ. Moreover, it is worth noting that $\zeta_{k,n}$ increases with k.

The loci curves of $M_{jk,1}(jz)$ in the complex plane are plotted in Fig. 4(a) and (b) for $k=+0.1$ and $k=-0.1$, respectively. The point representing $M_{jk,1}(jz)$ moves along a segment of the bisectrix of the second and fourth quadrants, starting from zero and passing through the origin at consecutive zeros $\zeta_{k,n}$ of the function. The middle point of the segment coincides with the point zero in the complex plane, its length being twice the modulus of the maximum of the Whittaker function $M_{jk,1}(jz)$.

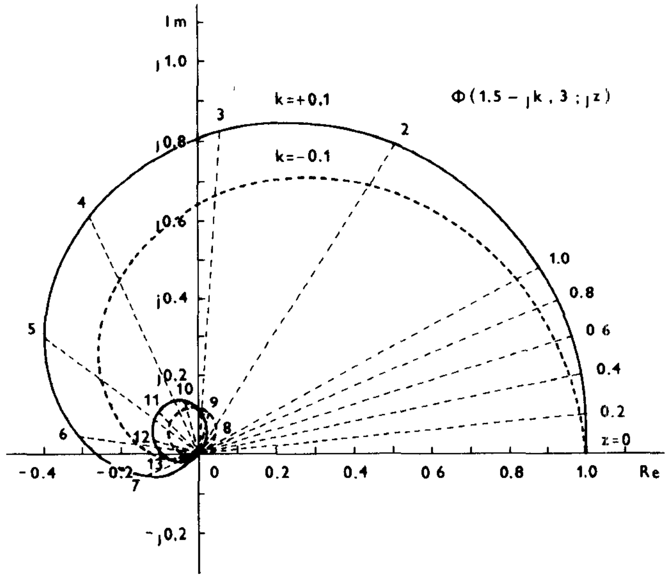


Fig. 2. Loci curves of $\Phi(1.5-jk,3;jz)$ in the complex plane for $k=\pm 0.1$.

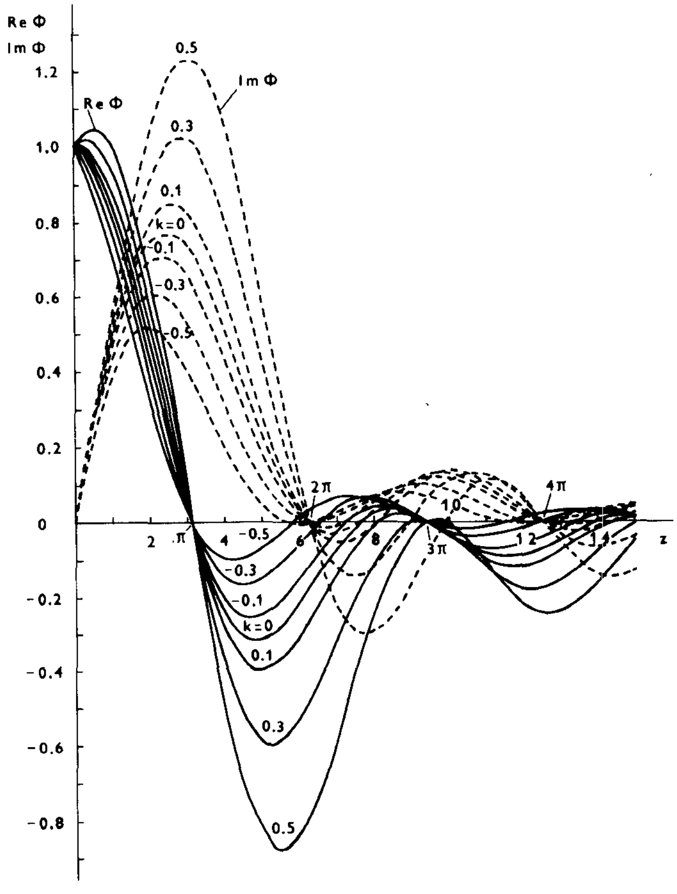
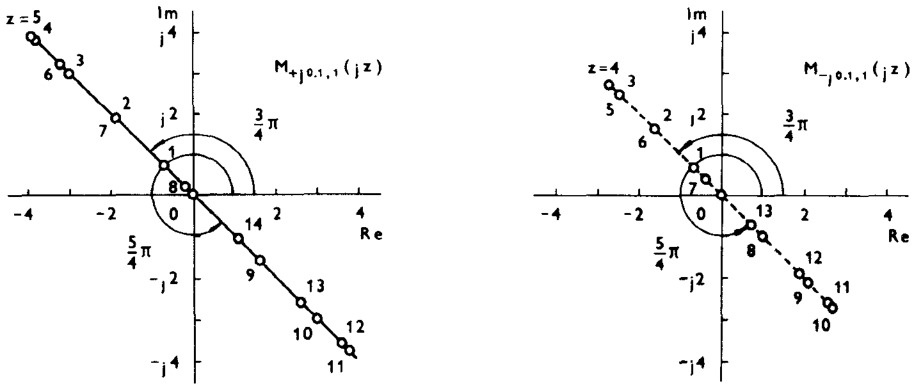


Fig. 3. Real and imaginary parts of the Kummer function $\Phi(1.5-jk,3;jz)$ versus z for $k=0,\pm 0.1,\pm 0.3,\pm 0.5$.

Fig. 5 shows the variation of $\text{Re}\,M_{jk,1}(jz)$ (solid curves) and $\text{Im}\,M_{jk,1}(jz)$ (dashed curves) versus z for $k = 0, \pm 0.1, \pm 0.3, \pm 0.5$. The points of intersection of both curves for the same k at the z-axis define the zeros $\zeta_{k,n}$ of the function. It should be emphasized that the real and imaginary parts of $M_{jk,1}(jz)$ exhibit perfect symmetry with respect to the horizontal axis.



(a) (b)

Fig. 4. Loci curves of the Whittaker first function $M_{jk,1}(jz)$. (a) $k=+0.1$ (b) $k=-0.1$.

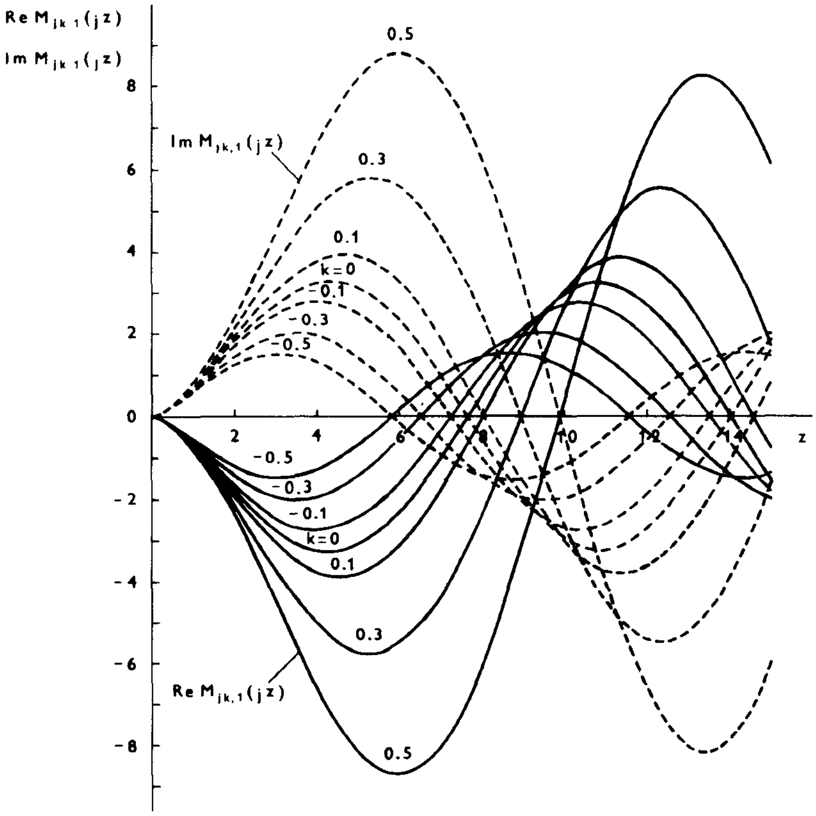


Fig. 5. Real and imaginary parts of the Whittaker first function $M_{jk,1}(jz)$ versus z for $k=0, \pm 0.1, \pm 0.3, \pm 0.5$.

The modulus and argument of the Kummer and Whittaker first function are plotted against the variable z in Figs. 6 and 7 for $k=\pm 0.1$. The analysis reveals that $\arg \Phi$ is a linear function of z with constant slope $1/2$ and discontinuities at each consecutive zero $\zeta_{k,n}$ of Φ where it increases abruptly by π. Arg M is a step function of z with initial value $(3/4)\pi$, having step increases by π at each consecutive zero $\zeta_{k,n}$ of M.

Referring to (17) or (21), it is seen that the factor $x^{(1/2)+m}e^{-x/2}$ introduced by Whittaker transforms the single-valued Kummer function $\Phi(a, c; jz)$ having simple and elegant properties into a multiple-valued function $M_{\kappa,m}(jz)$. This factor complicates the computations involving $M_{\kappa,m}(jz)$, but makes its main branch symmetrical, as evident from Figs. 4(a) and (b) and 5. This symmetry is of considerable importance in compiling tables of Whittaker first functions.

As mentioned above, the zeros of Kummer and Whittaker CHF coincide. Table I(a) lists the first and second imaginary zeros $\zeta_{k,1}$, $\zeta_{k,2}$ and Table I(b) lists the third and fourth zeros $\zeta_{k,3}, \zeta_{k,4}$ of $\Phi(1.5-jk,3; jz)$ and $M_{jk,1}(jz)$ to nine decimal places for k over the range $k=-1.0\ (0.1)+1.0$. A comparison of $\zeta_{0,1} \div \zeta_{0,4}$ with the 30 figure tabulated zeros of Bessel functions J_1 [18], taking into account the second Kummer theorem [16] (cf. the Appendix), gives an excellent accuracy. The distribution of the first four zeros $\zeta_{k,n}$ of Φ $(1.5-jk,3; jz)$, respectively, $M_{jk,1}(jz)$ versus k, is plotted in Fig. 8. The curves intersect

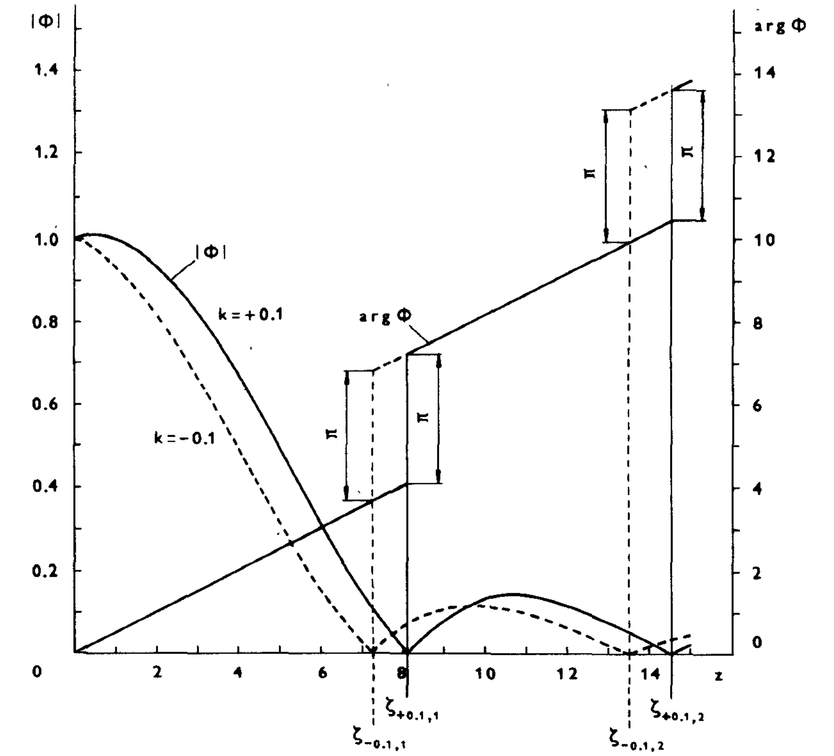


Fig. 6. Modulus and argument of the Kummer function $\Phi(1.5-jk,3;\, jz)$ versus z for $k=\pm 0.1$.

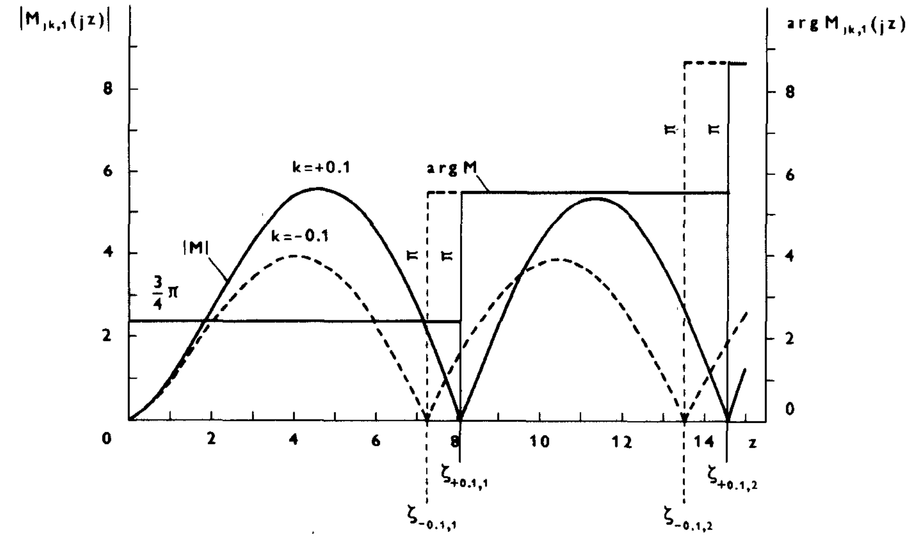


Fig. 7. Modulus and argument of the Whittaker first function $M_{jk,1}(jz)$ versus z for $k=\pm 0.1$.

the ordinate axis ($k=0$) at points $\zeta_{0,n}=2\nu_{1,n}$, $\nu_{1,n}$ being the nth consecutive zero of Bessel function J_1 (cf. (42) in the Appendix)

VII. Application of Wave Functions

As an example of how the wave functions studied in the previous sections can be applied, we shall take the case of the circular gyrotropic guide, filled with azimuthally magnetized remanent ferrite. We are interested in the phase characteristics of this structure. The relations $\bar{\beta}^2+\bar{\beta}_2^2=1-\alpha^2$, $k=\alpha\bar{\beta}/2\bar{\beta}_2$ and Table I(a) are used to compute the normalized phase characteristics $\bar{\beta}(\bar{r}_0)$ of the structure for the $\mathrm{TE}_{0,1}$ mode. The quantities $\bar{\beta}=\beta/\beta_0\sqrt{\epsilon_r}$, $\bar{\beta}_2=\beta_2/\beta_0\sqrt{\epsilon_r}$ and $\bar{r}_0=\beta_0 r_0\sqrt{\epsilon_r}=\zeta_{k,1}/2\bar{\beta}_2$ are the phase constant, radial wavenumber, and guide radius, normalized with the free space phase constant $\beta_0=\omega\sqrt{\epsilon_0\mu_0}=\omega/c$ and the square root of relative permittivity ϵ_r (c being light velocity in free space). The results are plotted in Fig. 9 for positive and negative directions of M_r (solid and dashed lines) with $p=\gamma M_r/\omega$ as parameter. The phase of the $\mathrm{TE}_{0,1}$ mode propagating along the guide can be shifted digitally by the reversal of M_r. The normalized differential phase shift $\Delta\bar{\beta}=\bar{\beta}_{-}-\bar{\beta}_{+}$ may easily be computed from

TABLE I

(a) First and Second Zeros of $\Phi(1.5-jk,3;jz)$ and $M_{jk,1}(jz)$

k	$\zeta_{k,1}$	$\zeta_{k,2}$
-1.0	4.475 056 705	9.577 795 687
-0.9	4.709 734 370	9.942 777 879
-0.8	4.961 885 767	10.325 939 326
-0.7	5.232 365 644	10.727 561 384
-0.6	5.521 865 663	11.147 732 639
-0.5	5.830 856 452	11.586 309 544
-0.4	6.159 534 497	12.042 886 465
-0.3	6.507 775 558	12.516 772 694
-0.2	6.875 107 390	13.006 989 713
-0.1	7.260 704 635	13.512 285 497
0.0	7.663 411 940	14.031 173 339
+0.1	8.081 794 373	14.561 989 239
+0.2	8.514 210 131	15.102 964 099
+0.3	8.958 896 447	15.652 301 729
+0.4	9.414 057 790	16.208 253 485
+0.5	9.877 945 308	16.769 181 612
+0.6	10.348 921 178	17.333 604 853
+0.7	10.825 503 272	17.900 228 114
+0.8	11.306 387 988	18.467 950 312
+0.9	11.790 459 616	19.035 863 241
+1.0	12.276 782 505	19.603 235 310

(b) Third and Fourth Zeros of $\Phi(1.5-jk,3;jz)$ and $M_{jk,1}(jz)$

k	$\zeta_{k,3}$	$\zeta_{k,4}$
-1.0	15.074 466 011	20.775 857 695
-0.9	15.527 595 295	21.294 022 412
-0.8	15.998 093 617	21.828 676 575
-0.7	16.486 001 509	22.379 746 636
-0.6	16.991 173 472	22.946 979 485
-0.5	17.513 244 559	23.529 911 221
-0.4	18.051 607 427	24.127 847 096
-0.3	18.605 395 505	24.739 847 206
-0.2	19.173 485 800	25.364 732 009
-0.1	19.754 518 280	26.001 103 039
0.0	20.346 936 270	26.647 384 457
+0.1	20.949 044 396	27.301 878 653
+0.2	21.559 079 516	27.962 839 912
+0.3	22.175 279 508	28.628 541 489
+0.4	22.795 962 249	29.297 357 807
+0.5	23.419 572 550	29.967 759 708
+0.6	24.044 726 293	30.638 462 659
+0.7	24.670 233 475	31.308 251 067
+0.8	25.295 100 809	31.976 379 688
+0.9	25.918 526 932	32.641 958 372
+1.0	26.539 884 234	33.304 553 267

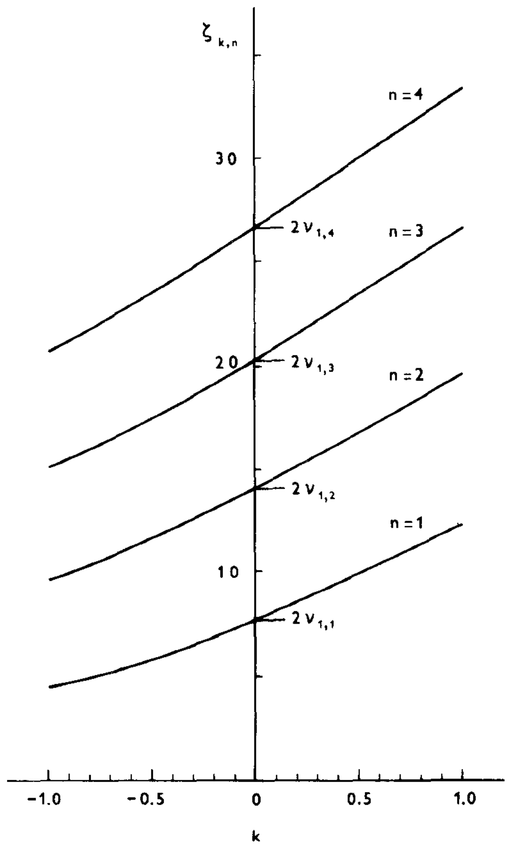


Fig. 8. Distribution of the first four imaginary zeros of $\Phi(1.5-jk,3;jz)$ and $M_{jk,1}(jz)$ with k.

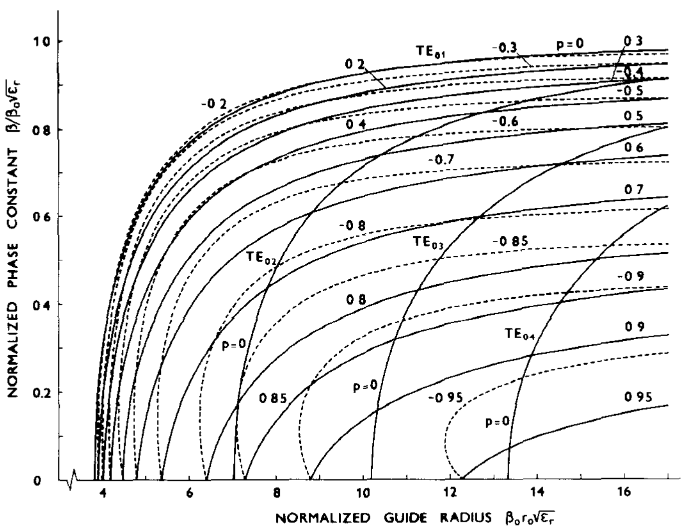


Fig. 9. Normalized phase characteristics of the azimuthally magnetized ferrite-loaded circular guide.

the characteristics. Thus, the circular gyrotropic guide appears to be a suitable device configuration for the single-bit ferrite remanent phaser.

At the bifurcation points of each two branches of phase characteristics for given values of p, $\bar{\beta}_+ = 0$ and $\bar{\beta}_- \neq 0$, i.e., the $TE_{0,1}$ mode propagation is possible for negative M_r only. This feature reveals the additional potentialities of the gyrotropic guide as a fast remanent cutoff switch with two stable operation states (on and off positions), corresponding to $-M_r$ and $+M_r$, respectively.

Also shown in the same figure for completeness are curves of $\bar{\beta}$ versus $\bar{r}_0$ of the homogeneously filled circular guide with isotropic dielectric of permittivity ϵ_r for the first four angular symmetric TE modes.

VIII. Conclusion

A class of electromagnetic wave functions for propagation along the circular guide, containing latching ferrite, magnetized azimuthally to remanence by a coaxially positioned switching conductor of finite radius, is presented. The straightforward solution of Maxwell equations yields a system of second-order partial differential equations for the longitudinal components of electric and magnetic vectors which shows the existence of a six-component electromagnetic wave in the anisotropic ferrite. In the degenerate case of angular symmetric fields, this system splits into two independent second-order ordinary differential equations describing TM and TE modes. The TM modes propagate along the gyrotropic structure as in a circular guide filled with isotropic dielectric. Of special interest is the equation for the longitudinal component of the magnetic field, which is found to be the Kummer or Whittaker form of the confluent hypergeometric equation involving ferrite magnetic parameters. Two ways of writing the TE field components are presented—in terms of Kummer and Tricomi CHF and in terms of Whittaker first and second functions. A profound discussion of the properties of these exact wave functions is performed, describing the nonreciprocal character of TE mode propagation in the gyrotropic guide. Two forms of the characteristic equation of the circular gyrotropic guide under angular symmetric TE modes excitation are derived.

Assuming an infinitely thin switching conductor, the structure is reduced to a ferrite-filled circular guide—a canonical structure for comprehensive study of the Kummer and Whittaker first function of complex parameter and variable and their eigenvalues. These wave functions are evaluated numerically, and tables of their imaginary zeros have been compiled for the first time. Based on the numerical and graphical results, conclusions are drawn concerning the behavior of this class of wave functions.

The normalized phase characteristics of the structure for the $TE_{0,1}$ mode, calculated by means of tabulated imaginary zeros of wave functions, are presented graphically for a variety of geometry and ferrite parameters. Two important effects in the gyrotropic guide—nonreciprocal phase shifting and magnetically controlled cutoff—experienced by the $TE_{0,1}$ mode are established, which can be used in designing ferrite control components for microwave frequencies.

The wave functions studied in this paper may be of interest in boundary-value analysis of guiding structures with circular symmetry, containing gyrotropic media magnetized azimuthally to the direction of wave propagation.

Appendix
Second Kummer Theorem

The Second Kummer theorem [16] gives the relationship between the CHF and cylindrical functions

$$\Phi(a,2a;\, j2x) = e^{jx}\left(\tfrac{1}{2}x\right)^{1/2-a}\Gamma\left(a+\tfrac{1}{2}\right)J_{a-1/2}(x) \quad (38)$$

$$\Psi(a,2a;\, j2x) = \tfrac{1}{2}\sqrt{\pi}\, e^{j(x-a\pi)}H^{(2)}_{a-1/2}(x)(2x)^{1/2-a} \quad (39)$$

$$M_{0,m}(\pm j2x) = \Gamma(1+m)e^{\pm j\pi/2(1/2+m)} \cdot 2^{2m+1/2}\cdot x^{1/2}J_m(x) \quad (40)$$

$$W_{0,m}(j2x) = \sqrt{\frac{\pi x}{2}}\, e^{-j\pi/2(1/2+m)}H^{(2)}_m(x) \quad (41)$$

where $H^{(2)}_{a-1/2} = H^{(2)}_m$ is an mth-order Hankel function of the second kind.

According to (38) and (40), the consecutive roots $\zeta_{0,n}$ $(n=1,2,3,\ldots)$ of $\Phi(1.5,3;\, j2x)$ and $M_{0,1}(j2x)$, and the roots $\nu_{1,n}$ of $J_1(x)$ are related by the expression

$$\zeta_{0,n} = 2\nu_{1,n}. \quad (42)$$

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